Problems for the week

Introductory Number Theory

a) Prove that for any integer n

 1) $n^3 - n$ is divisible by 3
 2) $n^5 - n$ is divisible by 5
 3) $n^7 - n$ is divisible by 7
 4) $n^{11} - n$ is divisible by 11
 5) $n^{13} - n$ is divisible by 13

 Do you notice a pattern? If so, form a conjecture of the general result. Try to prove the conjecture.

b) Tom multiplied two two-digit numbers on the blackboard. Then he changed all the digits to letters (different digits were changed to different letters, and equal digits were changed to the same letter). He obtained AB$\times$ CD = EEFF. Prove that Tom made a mistake somewhere.

c) The numbers a and b satisfy the equation $56a = 65b$. Prove that $a + b$ is composite.

d) What is the last digit of 777^{777}?

e) Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7. *Hint:* Show that the remainder when the given number is divided by 7 is zero.

Geometry

a) In $\Delta\ ABC$, D, E, F are points on sides BC, CA, AB. Also, X, Y, Z are points on FE, DF and DE of $\Delta\ DEF$ such that $AB \parallel XY$, $BC \parallel YZ$, and $AC \parallel XZ$. Prove that

$$\text{Area of } \Delta DEF = \sqrt{\text{Area of} of \Delta ABC \times \text{Area of} \Delta XYZ}$$

b) Prove that the area of triangle formed by the medians of ΔPQR is equal to three fourth the area of ΔPQR.

c) Prove that in any triangle, at most only one side can be shorter than the altitude from the opposite vertex.

d) (1.387) The point C lies inside the right angle AOB. Prove that the perimeter of triangle ABC is greater than $2 \times OC$.

e) (1.389) The point M is located inside the triangle ABC. Prove that $(BM + CM) < (AB + AC)$.