On Square Numbers[1]

This article is meant for use as enrichment material. Suggested levels of usage are indicated below.

One of the great pleasures of mathematics is the way that hidden connections crop up between objects or relationships, typically in an unexpected and counter-intuitive manner. For the teacher this has obvious pedagogic value, because to confront such a phenomenon, or to come upon it suddenly, is to feel astonished, delighted and intrigued all at the same time. A student who experiences this feeling may never feel the same way about the subject again. Of course, it is necessary for the teacher to do more than merely mystify; a serious attempt must be made to penetrate into the topic and to account for these hidden connections.

In this article we shall elaborate on these comments through an exploration of the sequence of square numbers,

$$\mathcal{S} = \{1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 144,\ 169,\ \ldots\}. \tag{1}$$

We shall explore four pretty and rather striking properties of the sequence, as listed below. Clicking on any of the four items will take you to the relevant section.

- Squares and odd numbers (appropriate for classes 8–10)
- Squares and divisors (appropriate for classes 8–11)
- Squares and multiples of 2 and 3 (appropriate for classes 8–11)
- Squares and cubes (appropriate for classes 11–12)

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Squares and odd numbers

Consider the sequence of odd numbers,

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ \ldots. \tag{2}$$

If we make a list of the *partial sums* of this sequence, in other words, the sums 1, $1+3$, $1+3+5$, $1+3+5+7$, $1+3+5+7+9$, $\ldots$, we get the following numbers:

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ \ldots. \tag{3}$$

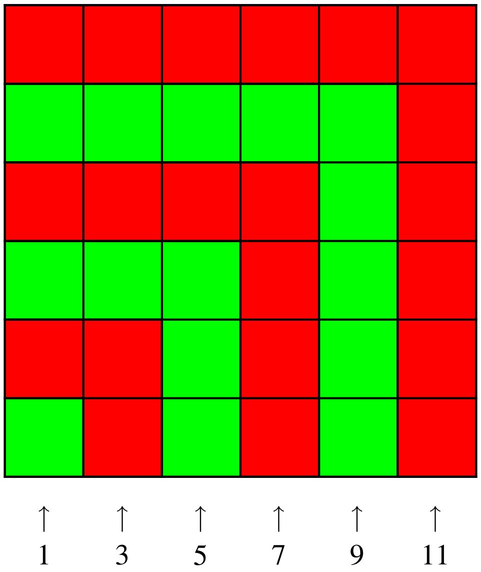
It seems clear that we get the sequence of squares, $\mathcal{S}$. In other words, we seem to have:

$$1+3+5+\cdots+(2k-1)=k^2, \tag{4}$$

for any positive integer k.

Proof

There are various algebraic and arithmetical ways of explaining why this is so, but the nicest is a pictorial approach.



In the above diagram, the strips alternately coloured green and red have 1, 3, 5, 7, 9, 11, . . . squares, respectively, and it is only natural that

$$1=1^2,\quad 1+3=2^2,\quad 1+3+5=3^2,\quad 1+3+5+7=4^2,\quad 1+3+5+7+9=5^2,$$

and so on. And that's the proof! ♣

Squares and divisors

Given any positive integer, let us count its total number of divisors, including the number itself. For example, take the number 20. Since the divisors of 20 are

$$1,\ 2,\ 4,\ 5,\ 10,\ 20,$$

we get the count as 6.

We now prepare a list of values of the number of divisors, displaying them as shown below.

n	Divisors of n	Number of divisors of n
1	1	1
2	1, 2	2
3	1, 3	2
4	1, 2, 4	3
5	1, 5	2
6	1, 2, 3, 6	4
7	1, 7	2
8	1, 2, 4, 8	4
9	1, 3, 9	3
10	1, 2, 5, 10	4
11	1, 11	2
12	1, 2, 3, 4, 6, 12	6
13	1, 13	2
14	1, 2, 7, 14	4
15	1, 3, 5, 15	4
16	1, 2, 4, 8, 16	5
17	1, 17	2
18	1, 2, 3, 6, 9, 18	6
19	1, 19	2
20	1, 2, 4, 5, 10, 20	6

We now focus on the *parity* of the entries in the third column; that is, on whether the entries are odd or even. We find that among the numbers from 1 till 20, the number of divisors is odd when $n = 1$, 4, 9 and 16.

These numbers are just the squares below 20! So we make a bold guess and put forward the following hypothesis: *The number of divisors of n is odd precisely when n is a square number*. It turns out that our guess is correct.

Proof

The proof is breathtakingly simple. Let n be any integer, and let S be the set of divisors of n, including n itself. We pair the divisors with one another as follows: if k is a divisor, we pair it with the divisor n/k. (It will be evident that if k is a divisor of n, then so is n/k. The two are called “complementary divisors”.)

Pairing the divisors in this manner, it becomes clear that their total number is an even number—*unless* there is a divisor that has not been able to find a “mate” and thus is left unpaired. This can happen if and only if k and n/k are the same, which is the same as saying that $k^2 = n$. This can happen only if n is a square number. It follows that it is precisely the square numbers that have an odd number of divisors.

An example may help clarify the argument. Consider the number 20. Its divisors are 1, 2, 4, 5, 10 and 20. These may be paired thus:

$$\{1,20\}, \quad \{2,10\}, \quad \{4,5\}.$$

Observe that each divisor has found its mate, so the number of divisors is even.

Similarly, for the number 30, whose divisors are 1, 2, 3, 5, 6, 10, 15 and 30, we get the following pairs:

$$\{1,30\}, \quad \{2,15\}, \quad \{3,10\}, \quad \{5,6\}.$$

As earlier, each divisor has found a mate.

But now consider the number 16, whose divisors are 1, 2, 4, 8 and 16. When we attempt to pair them up we get:

$$\{1,16\}, \quad \{2,8\} \quad \{4,?\}.$$

The divisor 4 is unable to find a mate, so 16 has an odd number of divisors. This will clearly be the case for any square number. ♣

Squares and multiples of 2 and 3

Take any multiple of 6, and enumerate all the numbers smaller than it that are either multiples of 2, *or* multiples of 3, but *not* of both. Find their sum, and then divide by 9. *The answer is always a square number!* Indeed, if the original number is $6k$, then the sum is equal to k^2.

For example, let $n = 12$, with $k = 2$. The numbers below 12 that are either multiples of 2 or multiples of 3, but not of both, are

$$2,\ 3,\ 4,\ 8,\ 9,\ 10.$$

Their sum is $2+3+4+8+9+10 = 36$, and $36 \div 9 = 4 = 2^2 = k^2$.

Or take $n = 18$, with $k = 3$. The numbers below 18 that are either multiples of 2 or multiples of 3, but not of both, are

$$2,\ 3,\ 4,\ 8,\ 9,\ 10,\ 14,\ 15,\ 16.$$

Their sum is $2+3+4+8+9+10+14+15+16 = 81$, and $81 \div 9 = 9 = 3^2 = k^2$.

Proof. We argue as follows.

- Among the numbers from 1 till 6, the numbers that are multiples of 2 or 3, but not both, are 2, 3 and 4, and the sum of these numbers is 9.
- Among the numbers from 7 till 12, the numbers that are multiples of 2 or 3, but not both, are 8, 9 and 10, and the sum of these numbers is $27 = 9 \times 3$.
- Among the numbers from 13 till 18, the numbers that are multiples of 2 or 3, but not both, are 14, 15 and 16, and the sum of these numbers is $45 = 9 \times 5$.

This progression continues, and it is easy to spot the general pattern. In the set of numbers from $6(k-1)+1$ till $6k$, the numbers that are multiples of 2 or 3, but not both, are

$$6(k-1)+2 = 6k-4, \quad 6(k-1) = 3 = 6k-3, \quad 6(k-1)+4 = 6k-2,$$

and their sum is $18k-9 = 9 \times (2k-1)$.

Therefore, the sum of all these numbers is

$$9+27+45+\cdots+(18k-9) = 9 \times [1+3+5+\cdots+(2k-1)] = 9k^2, \tag{5}$$

invoking what we already know (equation 4 on page 2). Therefore, sum $\div\, 9 = k^2$, and we have proved the claim. ♣

Squares and cubes

In this section, we compute the partial sums of the sequence of cubes,

$$1^3 = 1, \quad 2^3 = 8, \quad 3^3 = 27, \quad 4^3 = 64, \quad 5^3 = 125, \quad 6^3 = 216, \quad \dots. \tag{6}$$

We get the following.

$$\begin{aligned}
1^3 &= 1, \\
1^3+2^3 &= 9, \\
1^3+2^3+3^3 &= 36, \\
1^3+2^3+3^3+4^3 &= 100, \\
1^3+2^3+3^3+4^3+5^3 &= 225, \\
1^3+2^3+3^3+4^3+5^3+6^3 &= 441,
\end{aligned}$$

and so on. Looking closely at the results, we notice that the sums are all squares! Indeed, we find that

$$\begin{aligned}
1 &= 1^2 = 1^2, \\
9 &= 3^2 = (1+2)^2, \\
36 &= 6^2 = (1+2+3)^2, \\
100 &= 10^2 = (1+2+3+4)^2 , \\
225 &= 15^2 = (1+2+3+4+5)^2, \\
441 &= 21^2 = (1+2+3+4+5+6)^2,
\end{aligned}$$

and so on. Therefore, it seems to be the case that

$$1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2. \tag{7}$$

What a curious connection between the cubes and the squares!

Proof

The statement may be proved using induction. Since the sum $1+2+3+\cdots+n$ equals $n(n+1)/2$, the above claim may be written as:

$$1^3+2^3+3^3+\cdots+n^3 = \frac{n^2\cdot(n+1)^2}{4}. \tag{8}$$

If this statement is denoted by $\mathcal{P}_n$, then $\mathcal{P}_1$ is true by inspection, thus providing an anchor to the induction. Next, assuming that $\mathcal{P}_k$ is true for some integer k, we add $(k+1)^3$ to both sides of the assumed equality,

$$1^3+2^3+3^3+\cdots+k^3=\frac{k^2\cdot(k+1)^2}{4},$$

getting $1^3+2^3+3^3+\cdots+k^3+(k+1)^3$ on the left side, and on the right side,

$$\frac{k^2(k+1)^2}{4}+(k+1)^3=\frac{(k+1)^2}{4}\cdot\left(k^2+4(k+1)\right)=\frac{(k+1)^2\cdot(k+2)^2}{4},$$

implying that

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{(k+1)^2\cdot(k+2)^2}{4}.$$

This is just the statement $\mathcal{P}_{k+1}$.

So the inductive step is verified, and the claim (7) is proved. ♣

Is a pictorial proof possible for this identity? We leave this question to the reader.

A surprising extension

The above property and its proof will probably be known to any student who has studied the topic of mathematical induction at the "Plus Two" level. However, there is a surprising extension of the property that is not too well known.

Consider again the identity (7); it may be described differently, as follows.

> *The list of numbers* $1, 2, 3, \ldots, n-1, n$ *has the property that the sum of their cubes is equal to the square of their sum.*

We now show how to generate infinitely many lists of numbers with just this property. (We use the term 'list' rather than 'set' as there may be duplicate entries in the list.) In fact, we shall show how to generate such a list corresponding to each positive integer N. We illustrate the method with $N=10$ and $N=12$.

- $\boxed{N=10}$ We start by listing the divisors of 10; they are

$$1,\quad 2,\quad 5,\quad 10.$$

Next, for each of these divisors we list the number of divisors that *it* has; we get the list

$$1, \quad 2, \quad 2, \quad 4. \tag{9}$$

This is so because 1 has just one divisor, 2 and 5 have two divisors each, and 10 has four divisors. It will now be found that the list (9) we have obtained has the property claimed; for

$$\begin{aligned} &1^3+2^3+2^3+4^3=1+8+8+64=81, \\ &(1+2+2+4)^2=9^2=81. \end{aligned}$$

- $\boxed{N=12}$ We list the divisors of 12:

  $$1, \quad 2, \quad 3, \quad 4, \quad 6, \quad 12.$$

  These numbers have, respectively,

  $$1, \quad 2, \quad 2, \quad 3, \quad 4, \quad 6 \tag{10}$$

  divisors (1 has one divisor, 2 and 3 have two divisors each, 4 has three divisors, 6 has four divisors, and 12 has six divisors). So we get the list 1, 2, 2, 3, 4, 6. Now observe that

  $$\begin{aligned} &1^3+2^3+2^3+3^3+4^3+6^3=1+8+8+27+64+216=324, \\ &(1+2+2+3+4+6)^2=18^2=324. \end{aligned}$$

Why should this work for any N? We leave the proof to the reader.

Question. For what value of N will this method yield the identity (7)?